On Tarski's fixed point theorem

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Introduction

The fixed point theorem referred to in this paper is the one asserting that every monotone mapping on a complete lattice L has a least fixed point. The proof, due to A. Tarski, of this result, is a simple and most significant example of a proof that can be carried out on the base of intuitionistic logic (e.g. in topos logic), and that yet is widely regarded as essentially non-constructive.

The reason for this fact is that Tarski's construction of the fixed point is highly impredicative: if $f : L \to L$ is a monotone map, its least fixed point is given by $\bigwedge P$, with $P \equiv \{x \in L \mid f(x) \leq x\}$. Impredicativity here is found in the fact that the fixed point, call it p, appears in its own construction (p belongs to P), and, indirectly, in the fact that the complete lattice L (and, as a consequence, the collection P over which the infimum is taken) is assumed to form a set, an assumption that seems only reasonable in an intuitionistic setting in the presence of strong impredicative principles (cf. Section 2 below).

Alternative, more constructive, proofs of this result have been proposed, probably the two most satisfactory so far being those obtained in intuitionistic contexts in [14, 20]. These proofs, however, still presuppose the existence

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of powerobjects (use the powerset axiom), and/or make use of fully impredicative comprehension principles.

In this paper I present a constructive and predicative version of Tarski's fixed point theorem. Working in the context of the constructive set theories initiated by John Myhill and Peter Aczel, I formulate a concept of *abstract inductive definition* on a complete lattice, and use it to obtain a generalization to complete lattices of Aczel's theory of inductive definitions on a set. Every abstract inductive definition gives rise to an inductively defined 'generalized element' of the lattice. Using this fact, I first derive a proof of Tarski's theorem in a basic system for constructive set theory extended by the full Separation scheme. This proof may be regarded as an improvement on the mentioned results in [14, 20]. Then, under assumptions on the lattice and the monotone map that are always satisfied in fully impredicative (classical or intuitionistic) systems, a constructive and predicative proof of Tarski's theorem is obtained.

Before discussing (respectively in sections 3 and 4) abstract inductive definitions and their application to the fixed point theorem, I formulate in Section 2 a notion of *uniform class*, by analogy with the concept of uniform object studied in the context of the effective topos. This notion is used proof-theoretically to show that in the systems for constructive set theory we work with, certain standard partially ordered structures, as complete lattices or directed-complete partial orders, must be defined as having a proper class of elements.

1 Constructive set theory

I shall be working in the setting of Aczel-Myhill's constructive set theories. The basic system in this context is the choice-free *Constructive Zermelo-Fraenkel Set Theory* (CZF), due to Aczel. This system is often extended by constructively acceptable axioms of choice, as Dependent Choice or the Presentation Axiom, and with principle, such as the Regular Extension Axiom, that ensure that certain inductively defined classes are sets. I here provide the basic information that make the paper self-contained; the reader may consult [5] for a thorough introduction to the subject.

The language of CZF is the same as that of Zermelo-Fraenkel Set Theory, ZF, with $\in$ as the only non-logical symbol. Beside the rules and axioms of a standard calculus for intuitionistic predicate logic with equality, CZF has

the following axioms and axiom schemes:

1. Extensionality: $\forall a \forall b (\forall y (y \in a \leftrightarrow y \in b) \rightarrow a = b)$.

2. Pair: $\forall a \forall b \exists x \forall y (y \in x \leftrightarrow y = a \vee y = b)$.

3. Union: $\forall a \exists x \forall y (y \in x \leftrightarrow (\exists z \in a)(y \in z))$.

4. Restricted Separation scheme:

$$\forall a \exists x \forall y (y \in x \leftrightarrow y \in a \wedge \phi(y)),$$

for ϕ a restricted formula. A formula ϕ is *restricted* if the quantifiers that occur in it are of the form $\forall x \in b$, $\exists x \in c$.

5. Subset Collection scheme:

$$\begin{aligned} &\forall a \forall b \exists c \forall u ((\forall x \in a)(\exists y \in b)\phi(x, y, u) \ \rightarrow \\ &(\exists d \in c)((\forall x \in a)(\exists y \in d)\phi(x, y, u) \wedge (\forall y \in d)(\exists x \in a)\phi(x, y, u))). \end{aligned}$$

6. Strong Collection scheme:

$$\begin{aligned} &\forall a ((\forall x \in a) \exists y \phi(x, y) \ \rightarrow \\ &\quad \exists b ((\forall x \in a)(\exists y \in b)\phi(x, y) \wedge (\forall y \in b)(\exists x \in a)\phi(x, y))). \end{aligned}$$

7. Infinity: $\exists a (\exists x \in a \wedge (\forall x \in a)(\exists y \in a) x \in y)$.

8. Set Induction scheme: $\forall a ((\forall x \in a)\phi(x) \rightarrow \phi(a)) \rightarrow \forall a \phi(a)$.

We shall denote by CZF^- the system obtained from CZF by leaving out the Subset Collection scheme. Note that from Subset Collection one proves that the class of functions b^a from a set a to a set b is a set, i.e., Myhill's Exponentiation Axiom.

Friedman's Intuitionistic Zermelo-Fraenkel set theory based on collection, IZF, has the same theorems as CZF extended by the unrestricted Separation Scheme and the Powerset Axiom. Moreover, the theory obtained from CZF, or from IZF, by adding the Law of Excluded Middle has the same theorems as ZF.

As in classical set theory, one makes use in this context of class notation and terminology [5]. A major role in constructive set theory is played by

inductive definitions. An *inductive definition* is any class Φ of pairs. A class A is $\Phi-$*closed* if:

$$(a, X) \in \Phi, \text{ and } X \subseteq A \text{ implies } a \in A.$$

The following theorem is called *the class inductive definition theorem* [5].

Theorem 1.1 (CZF$^-$) *Given any class Φ, there exists a least $\Phi-$closed class $I(\Phi)$, the class inductively defined by Φ.*

An *inductive definition on a set* is an inductive definition Φ that is a subclass of the cartesian product $S \times \mathsf{Pow}(S)$, for S a set ($\mathsf{Pow}(S)$ is the class of subsets of S).

Even when Φ is a set, $I(\Phi)$ need not be a set in CZF. For this reason, CZF is often extended with the *Regular Extension Axiom*, REA.

REA: every set is the subset of a regular set.

A set c is *regular* if it is transitive, inhabited, and for any $u \in c$ and any set $R \subseteq u \times c$, if $(\forall x \in u)(\exists y)\langle x, y\rangle \in R$, then there is a set $v \in c$ such that

$$(\forall x \in u)(\exists y \in v)((x, y) \in R) \quad \wedge \quad (\forall y \in v)(\exists x \in u)((x, y) \in R). \qquad (1)$$

c is said to be *weakly regular* if in the above definition of regularity the second conjunct in (1) is omitted. The weak regular extension axiom, wREA, is the statement that every set is the subset of a weakly regular set.
In CZF + wREA, the following theorem can be proved.

Theorem 1.2 (CZF + wREA) *If Φ is a set, then $I(\Phi)$ is a set.*

The foregoing result holds in more generality for inductive definitions that are *bounded* (see [5]). A strengthening of REA that is sometimes considered, and that will be exploited in this paper, is the axiom uREA [4]:

uREA: every set is the subset of a union-closed regular set.

A regular set A is *union-closed* if whenever $a \in A$ then also $\bigcup a \in A$.

Sometimes one considers extensions of CZF by constructively acceptable choice principles, such as the principle of Countable Choice AC_ω, Dependent Choice DC, the Presentation Axiom, or the principle of Relativized Dependent Choice RDC. See [5] for their description. Note that PA implies DC which in turn implies AC_ω.

2 The uniform objects in constructive set theory

The notion of uniform object was introduced in connection with the effective topos (cf. [12, 18]). In the context of (constructive) set theory, we shall say that a class A is *uniform* if, for every set a, and every formula $\phi(x, y)$,

$$(\forall x \in A)(\exists y \in a)\phi(x, y) \to (\exists y \in a)(\forall x \in A)\phi(x, y).$$

We shall abbreviate this schema as $UO(A)$. Note that if A is a uniform object, and $f : A \to a$, with a any set, is a function, then f must be constant.

Every singleton set (terminal object) is uniform. If a set is not a singleton, it is not uniform, as follows logically by the definition of uniformity: define a set c *trivial* if it is a singleton, i.e. if $(\exists x \in c)(\forall y \in c)(x = y)$; c is *non-trivial* if it is not trivial. Note that the empty set is non-trivial.

Proposition 2.1 *Every uniform set is trivial.*

Proof. Assume A is a uniform class, and that there is a set c with $c = A$. Then, by the scheme $UO(A)$, for $\phi(x, y) \equiv x = y$, and $a = c$, one gets $(\exists x \in c)(\forall y \in c)(x = y)$.

This proposition holds (in particular) for CZF and every consistent extension of it. In fact, CZF, as well as any extension of CZF compatible with the law of excluded middle LEM, cannot prove that a non-trivial class A is uniform: assume LEM, and let A be any non-trivial class. Then, either $A = \emptyset$, which is non-uniform, or A has at least two elements a, b. In the latter case, one may define a non-constant function from A to $\{0, 1\}$, sending a to 0, and every element of A different from a to 1.

However, in extensions of CZF that are *not* consistent with LEM, one does find non-trivial uniform classes (as shown, even in these extensions, one will not find non-trivial uniform *sets*). Recall that CZF is consistent with the conjunction of the following two non-classical principles.

Troelstra's uniformity principle, **UP**: for every ϕ,

$$(\forall x)(\exists y \in \mathbb{N})\phi(x, y) \to (\exists y \in \mathbb{N})(\forall x)\phi(x, y).$$

Every set is subcountable, **SC**:

$(\forall x)(\exists U \in \mathsf{Pow}(\mathbb{N}))(\exists f) f : U \twoheadrightarrow x.$

Using these two principles one finds a first interesting uniform class.

Proposition 2.2 (CZF+UP+SC) *The universal class* $\mathrm{V} = \{x \mid x = x\}$ *is uniform, i.e.,* $UO(\mathrm{V})$.

Uniformity of the universal class can also be regarded as a principle in itself, the *generalized uniformity principle* (GUP). In [8, 9] we exploited this general form of the uniformity principle for proving the independence from constructive set theory of various (classically and) intuitionistically valid results.

Models of CZF (and of several of its extensions, including the system CZF + Sep + REA + PA), that also validate the conjunction of UP and SC, have been exhibited by various authors [6, 15, 16, 17, 19]. In the following, $\mathrm{CZF}^\sharp$ denotes any possible extension of CZF that is simultaneously consistent with SC and UP, and so with GUP.

As for the uniform objects in the effective topos (cf. [18]), we have:

Lemma 2.3 *If* X *is uniform, and* $f : X \twoheadrightarrow Y$ *is an onto mapping, then* Y *is uniform (and therefore a proper class if non-trivial).*

Proof. Easy calculations.

Using this lemma, we may find new interesting uniform classes.

Corollary 2.4 *CZF + UP + SC proves that for every set* X, $\mathsf{Pow}(X)$ *is uniform. The same system plus Sep (full Separation) proves that* $\mathsf{Pow}(X)$ *is uniform for every class* X.

Proof. The map $f : \mathrm{V} \to \mathsf{Pow}(X)$, $f(y) = y \cap X$ is onto. If X is a class, by Sep one gets that $y \cap X$ is a set and is therefore in $\mathsf{Pow}(X)$.

One might be tempted to think that uniformity of an object is connected with its size, and that in particular, if an injection $X \hookrightarrow Y$ of a uniform object X into Y exists, then Y is uniform. This is not so, as one may realize considering the class $\mathsf{Pow}(\{0\}) \cup \{*\}$, with $*$ an element not belonging to $\mathsf{Pow}(\{0\})$: clearly a non-constant function can be defined on this class.

A partially ordered class, or *poclass*, $(X, \leq)$ is a class X together with a class-relation $\leq$ that is reflexive, transitive, and antisymmetric. X is *flat* if $(\forall x, y \in X)(x \leq y \rightarrow x = y)$.

A partially ordered class $(X, \leq)$ is a (large) $\bigvee$-*semilattice* if every subset has a supremum; $(X, \leq)$ is a (large) $\bigwedge$-*semilattice* if it has infima of arbitrary subsets (note: a large $\bigvee$-semilattice need not be a $\bigwedge$-semilattice, nor conversely). A poclass is *directed complete*, or a (large) *dcpo*, if it has joins of *directed* subsets (a subset U of a poclass X is directed if it is inhabited, and whenever $x, y \in U$ there is $z \in U$ with $x, y \leq z$). A poclass is *conditionally complete*, or a (large) *bcpo* if every *inhabited* and *bounded* subset has a join. Finally, a partially ordered class is *chain-complete*, or a (large) *ccpo*, if every chain (i.e., totally ordered subset) has a join.

Our main application of the concept of uniform class is that, whenever non-flat, these structures cannot be carried by sets in CZF$^\sharp$. Note that for $\bigvee$-semilattices, $\bigwedge$-semilattices and chain-complete poclasses, to be non-flat it suffices to be non-trivial, as these structures always posses a bottom, or top, element. Part of the following result is proved in [9]; the proof to be presented gives in particular a more uniform explanation of the results obtained there.

Theorem 2.5 *The following partially ordered structures are carried by proper classes in CZF + SC + UP, and thus cannot be proved to be carried by sets in CZF, and its extensions CZF$^\sharp$:*

1. *Non-trivial $\bigvee$-semilattices and $\bigwedge$-semilattices.*

2. *Non-flat dcpo's; in particular, non-trivial dcpo's with a top or a bottom element.*

3. *Non-flat bcpo's; in particular, non-trivial bcpo's with a top or a bottom element.*

4. *Non-trivial chain-complete partially ordered classes.*

Proof. 1. The result for $\bigvee$-semilattices is proved in [9]. Note also that a $\bigvee$-semilattice is a chain-complete partially ordered class, and, when non-trivial, a non-flat dcpo and bcpo. The result holds dually for $\bigwedge$-semilattices.

2. Let D be a non-flat dcpo, and let $b \leq a$, for $a, b \in D$. Assume D is a set. Then, by Restricted Separation, the class

$$U_{a,b}(y) \equiv \{x \in D \mid x = a \ \& \ \emptyset \in y\} \cup \{b\}$$

is a set, and is directed, for every $y \in \mathrm{V}$. One may therefore consider the class

$$K_{a,b} = \{U_{a,b}(y) \mid y \in \mathrm{V}\}.$$

Trivially, $K_{a,b}$ is a uniform object, as the map sending $y \in \mathrm{V}$ to $U_{a,b}(y)$, is onto. Then, again by the assumption that D is a set, $\bigvee : K_{a,b} \to D$ has to be constant. Therefore $a = b$ (first take $y = \emptyset$, then $y = \{\emptyset\}$), and this for every $b \leq a$, against non-flatness.

3. The claim is proved as for the previous case, since $U_{a,b}(y)$ is also bounded.

4. Let C be non-trivial and chain-complete. C has bottom $\bot = \bigvee \emptyset$. Assume C is a set. Considering, for $z \in C$, the uniform class of chains

$$K_z = \{W_z(y) \mid y \in \mathrm{V}\},$$

with

$$W_z(y) \equiv \{x \in C \mid x = z \ \&\ \emptyset \in y\}$$

one may easily conclude. Alternatively, one could observe that $U_{a,b}(y)$ is a chain for every a, b, and that a non-trivial ccpo is non-flat.

Note that $\mathbb{N}$ with the flat order $=$ is a dcpo and a bcpo (so that, requiring in 2, 3, the weaker condition that the dcpo's, bcpo's are non-empty (or even inhabited) and non-trivial is not enough), and that $\mathbb{N}$ with the same order and a bottom (or a top) element added is not a dcpo constructively.

Remark 2.6 Assuming also Sep, we may improve on 1. We may show indeed that any $\bigvee$-semilattice (or $\bigwedge$-semilattice) L is actually a uniform object: $\bigvee : \mathsf{Pow}(L) \to L$ is onto, so that, by Lemma 2.3 and Corollary 2.4, L is uniform. Similarly for $\bigwedge$-semilattices. The same holds without Sep for so-called set-generated $\bigvee$-semilattices (see the next section): if B is a generating set for L, the map $\bigvee : \mathsf{Pow}(B) \to L$ is again onto.

By this result, various types of partially ordered structures that in particular are structures of the kinds contemplated by Theorem 2.5, fail to be carried by sets in CZF, and in every extension $\mathrm{CZF}^{\sharp}$: these include various types of domains considered in the denotational semantics of programming languages, frames (locales), preframes, continuous lattices (appropriately redefined), etc.

A further important discrepancy with the situation in fully impredicative, classical or intuitionistic, systems concerns the size of hom-objects in categories of these structures. The following corollary extends to dcpo's with $\perp$, often called domains or cpo's, a corresponding fact proved for $\bigvee$-semilattices and pre-frames in [10].

Corollary 2.7 *In the category Dom of domains, no non-trivial $Hom(X, Y)$ can be proved to be set-indexed in CZF$^\sharp$.*

Proof. This result is essentially proved showing that, ordered 'pointwise', $Hom(X, Y)$ is itself a domain. A difficulty is that, in general, an homomorphism in $Hom(X, Y)$ is itself a proper class, so that Theorem 2.5 cannot be directly applied. One may deal with this difficulty as is done in [10] for the case of $\bigvee$-semilattices and pre-frames.

Fortunately, a similar corollary does *not* obtain for frames: there are full subcategories of the categories of frames that are locally small. This fact allows for constructive predicative proofs of important theorems in topology [9, 10].

In topos theory, and sometimes in constructive mathematics, one considers the *MacNeille reals* R_m (also called *extended reals* by Troelstra), see e.g. [13]. R_m is a conditionally complete class, and, in a topos, or in IZF, is a set.

Corollary 2.8 *R_m, as any order-complete extension X of $\mathbb{Q}$, cannot be proved to form a set in CZF, and in every extensions CZF$^\sharp$.*

Remark 2.9 As remarked in 2.6, $\bigvee$-semilattices and $\bigwedge$-semilattices are uniform objects (in CZF+Sep+UP+SC). This is more generally the case for every structure $(A, f, ...)$, with at least one mapping $f : \mathsf{Pow}(A) \to A$ that is onto. By contrast, dcpo's, bcpo's may not be uniform, even when they are non-flat:

Fact. The class $(\mathsf{Pow}(\{0\}) \cup \{*\}, \leq)$, where $\leq$ is defined extending the inclusion relation on $\mathsf{Pow}(\{0\})$ by letting $* \leq *$, is a non-flat dcpo and bcpo.

As noted previously, CZF proves that $\mathsf{Pow}(\{0\}) \cup \{*\}$ is not uniform. This also implies that, by contrast with e.g. powerclasses, the class of directed sets of a poclass may not be uniform (as, otherwise, $\mathsf{Pow}(\{0\}) \cup \{*\}$ would be uniform, being the join of directed sets an onto map).

3 Abstract inductive definitions

In this and the next section abstract inductive definitions are introduced and used to extend the theory of inductive definitions on a set described in [2, 3, 5] (see also Section 1) to $\bigvee$-semilattices. In this section we shall be working in CZF^-.

By Theorem 2.5, in constructive set theory it is of no use to consider $\bigvee$-semilattices carried by sets. The standard counterpart of the classical notion of $\bigvee$-semilattice is in this context given by the concept of *set-generated* $\bigvee$-*semilattice* [5]. A (large) $\bigvee$-semilattice L is said to be set-generated if it has a generating set B, i.e. a subset B of L such that, for all $x \in L$,

i. $\downarrow^B x \equiv \{b \in B \mid b \leq x\}$ is a set,

ii. $x = \bigvee \downarrow^B x$.

The powerclass $\mathsf{Pow}(X)$ of a set X, ordered by inclusion, is a prototypical example of a set-generated $\bigvee$-semilattice, with generating set $B = \{\{x\} : x \in X\}$. Note that a set-generated $\bigvee$-semilattice is also a complete lattice.

Let L be a set-generated $\bigvee$-semilattice L with generating set B. An *abstract inductive definition on* L (in the following often just an inductive definition) is any class of ordered pairs $\Phi \subseteq B \times L$.

To define what it means for a subclass of the generating set B to be $\Phi-$closed we need the following notion. A subclass $Y \subseteq B$ will be called c_L-*closed* if, for every subset U of Y, the set $\downarrow^B \bigvee U$ is contained in Y; i.e., Y is c_L-closed if

$$\bigcup_{U \in \mathsf{Pow}(Y)} \downarrow^B \bigvee U = Y.$$

A c_L-closed class can be thought of as denoting a generalized element of L[1]. Note that if Y is a set, Y is c_L-closed iff $Y = \downarrow^B \bigvee Y$.

A class $Y \subseteq B$ will be said $\Phi-$*closed* if it is c_L-closed and, whenever $(b, a) \in \Phi$,

$$\downarrow^B a \subseteq Y \implies b \in Y.$$

We shall denote by $\mathcal{I}(\phi)$ the least $\Phi-$closed class, if it exists.

[1]When L is a complete Heyting algebra, such generalized elements also arise as generalized truth values in (predicatively defined) cHa-models for constructive set theory, cf. [11].

Given an abstract inductive definition Φ on L, and an element a in L, the class

$$\{b \in B \mid (\exists a')\ (b, a') \in \Phi \ \&\ a' \leq a\}$$

may not be a set in general. If for, every $a \in L$, this class is a set we say that Φ is *local*. A local abstract inductive definition Φ determines a mapping $\Gamma_\Phi : L \to L$, given by, for $a \in L$,

$$\Gamma_\Phi(a) \equiv \bigvee\{b \in B \mid (\exists a')\ (b, a') \in \Phi \ \&\ a' \leq a\}.$$

Note that if $a_1 \leq a_2$, then $\Gamma_\Phi(a_1) \leq \Gamma_\Phi(a_2)$, i.e. Γ_Φ is *monotone*.

Any monotone operator on L can in fact be obtained in this way from a local abstract inductive definition.

Proposition 3.1 *Let $\Gamma : L \to L$ be a monotone operator on L. Then, there is a local abstract inductive definition Φ_Γ such that, for every $a \in L$, $\Gamma(a) = \Gamma_{\Phi_\Gamma}(a)$.*

Proof. Define $\Phi_\Gamma \subseteq B \times L$ by

$$(b, a) \in \Phi_\Gamma \iff b \leq \Gamma(a).$$

Φ_Γ is local as, for $a \in L$, $\{b \in B \mid (\exists a')\ (b, a') \in \Phi_\Gamma \ \&\ a' \leq a\} = \{b \in B \mid (\exists a')\ b \leq \Gamma(a') \ \&\ a' \leq a\}$. By monotonicity of Γ, this class is the same as $\{b \in B \mid b \leq \Gamma(a)\}$, that is a set by the assumption that L is set generated. The join of this set therefore exists, and again as B is a set of generators, is equal to $\Gamma(a)$.

We presented the above simple proof in detail with the purpose of emphasizing the role of the assumption that L is set-generated. The following proposition explains the relationship between $\Phi-$closed *sets* of a local inductive definition, and (pre-) fixed points of the associated monotone operator.

Proposition 3.2 *Given a local inductive definition Φ on a $\bigvee$-semilattice L with generating set B, a one-to-one correspondence exists between the $\Phi-$closed subclasses of B that are sets, and the elements a of L such that $\Gamma_\Phi(a) \leq a$. Moreover, whenever the class $\mathcal{I}(\Phi)$ exists and is a set, Γ_Φ has a least fixed point.*

Proof. Assume $Y \subseteq B$ is $\Phi-$closed and that it is a set. Then, $\bigvee Y$ exists in L and we have:

$$\Gamma_\Phi(\bigvee Y) = \bigvee\{b \in B \mid (\exists a)\ (b,a) \in \Phi \ \&\ a \leq \bigvee Y\}.$$

To conclude that $\Gamma_\Phi(\bigvee Y) \leq \bigvee Y$, let $b \in B$ be such that there is $a \in L$ with $(b,a) \in \Phi$ and $a \leq \bigvee Y$. To show that $b \leq \bigvee Y$ it suffices to prove that $\downarrow^B a \subseteq Y$, since Y is $\Phi-$closed. But this follows, by $a \leq \bigvee Y$, from the assumption that Y is a set and that it is c_L-closed (take $U = Y$ in the definition of c_L-closed).

Conversely, to $a \in L$ such that $\Gamma_\Phi(a) \leq a$, we associate the c_L-closed class $\downarrow^B a$. As L is set-generated, $\downarrow^B a$ is a set; using the assumption that $\Gamma_\Phi(a) \leq a$, one immediately sees that $\downarrow^B a$ is also Φ-closed.

Finally, one has $Y = \downarrow^B \bigvee Y$, as Y is c_L-closed, and $a = \bigvee \downarrow^B a$, since L is set-generated.

Now assume $\mathcal{I}(\Phi)$ is a set. As $\mathcal{I}(\Phi)$ is $\Phi-$closed, by what has just been shown, $\Gamma_\Phi(\bigvee \mathcal{I}(\Phi)) \leq \bigvee \mathcal{I}(\Phi)$. To prove the converse, note that by monotonicity of Γ_Φ, $\Gamma_\Phi(\Gamma_\Phi(\bigvee \mathcal{I}(\Phi))) \leq \Gamma_\Phi(\bigvee \mathcal{I}(\Phi))$. Then, $\downarrow^B \Gamma_\Phi(\bigvee \mathcal{I}(\Phi))$ is $\Phi-$closed, by the first part of this proposition. Then, as $\mathcal{I}(\Phi)$ is the least $\Phi-$closed class, $\mathcal{I}(\Phi) \subseteq \downarrow^B \Gamma_\Phi(\bigvee \mathcal{I}(\Phi))$, so that $\bigvee \mathcal{I}(\Phi) \leq \Gamma_\Phi(\bigvee \mathcal{I}(\Phi))$. Thus $\bigvee \mathcal{I}(\Phi)$ is a fixed point for Γ_Φ. If $a \in L$ is another fixed point, then in particular $\Gamma_\Phi(a) \leq a$. Therefore $\downarrow^B a$ is $\Phi-$closed, and $\mathcal{I}(\Phi) \subseteq \downarrow^B a$, which gives $\bigvee \mathcal{I}(\Phi) \leq a$.

4 Tarski's fixed point theorem

Recall by the preliminaries that, by results in [2, 5], for any (standard) inductive definition $\Phi \subseteq X \times \mathsf{Pow}(X)$, for X any set, the least $\Phi-$closed class $I(\Phi)$ exists in the system CZF^-. In this section we prove that more generally the least (c_L-closed and) $\Phi-$closed class $\mathcal{I}(\Phi)$ exists for every abstract inductive definition Φ on a set-generated $\bigvee$-semilattice L.

As a corollary of this result we shall have that CZF^- extended by the full Separation scheme proves Tarski's fixed point theorem. This improves on previous intuitionistic proofs of the theorem, as those in [14] or [20].

Albeit 'more constructive' than those proofs[2], this result can hardly be

[2]Note that this result will already tell us that we will not be able to refute Tarski's fixed point theorem using the consistency of constructive set theory with the principles (G)UP and SC.

considered satisfactory. For the case of monotone operators on $\bigvee$-semilattices of the form $\mathsf{Pow}(X)$ for X a set, it directly follows by the results in [2, 3, 5] that a restricted version of Tarski's theorem obtains (in fact a version of Knaster's theorem): if a monotone operator $\Gamma : \mathsf{Pow}(X) \to \mathsf{Pow}(X)$ may be obtained as Γ_Φ by a bounded inductive definition Φ (in particular by an inductive definition Φ that is a set), then the system CZF + REA proves that a least fixed point exists, as it proves that $I(\Phi)$ is a set in this case.

Here we show that, in CZF extended by the axiom uREA, a bounded abstract inductive definition on a set-generated $\bigvee$-semilattice L gives rise to a least $\Phi-$closed class $\mathcal{I}(\Phi)$ that is a set, whenever L satisfies the extra standard condition of being set-presented. In a sense there is no restriction here: in (I)ZF say, every abstract inductive definition is a set (and is therefore bounded), and every complete lattice is set-presented.

We now prove that for every abstract inductive definition Φ on L the least (c_L-closed and) $\Phi-$closed class $\mathcal{I}(\Phi)$ exists. The classical proof would construct $\mathcal{I}(\Phi)$ iterating a monotone operator associated with Φ along the class of ordinals. For the case of monotone operators on point-classes, Aczel [2] showed that one can replace the class of ordinals with the class of all sets (applications of transfinite induction are then replaced by applications of Set Induction). Here we follow the same approach.

The proof that the least $\Phi-$closed class $\mathcal{I}(\Phi)$ exists is a generalization of the proof in [2, 5] that the least $\Phi-$closed class exists for every standard inductive definition $\Phi \subseteq X \times \mathsf{Pow}(X)$, for X a set. Due to the 'pointless' nature of $\bigvee$-semilattices some difficulties however arise; the generalization in particular will require extra applications of the Strong Collection scheme. To make visible what is involved in the generalization, we shall follow the terminology and notation of [2, 5] as far as possible; we shall however use suggestively Greek letters for sets playing the role of ordinals.

Let B be a generating set for L and Φ an inductive definition on L. Given a class of ordered pairs $Y \subseteq \mathrm{V} \times B$, let, for α any set in the universal class V:

$$Y^\alpha \equiv \{y \in B \mid (\alpha, y) \in Y\};$$

$$Y^{\in\alpha} \equiv \{y \in B \mid \exists \beta \in \alpha\ (\beta, y) \in Y\} \equiv \textstyle\bigcup_{\beta\in\alpha} Y^\beta;$$

$$Y^\infty \equiv \{y \in B \mid \exists \alpha \in \mathrm{V}\ (\alpha, y) \in Y\} \equiv \textstyle\bigcup_{\alpha\in\mathrm{V}} Y^\alpha.$$

We shall need to consider an extension of the operator Γ_Φ. To this aim, we define an operator on subclasses of B which has the properties of a closure operator. For Y a subclass of B, let

$$c_L Y \equiv \{b \in B \mid (\exists U \in \mathsf{Pow}(B)) U \subseteq Y \ \& \ b \in {\downarrow^B} \bigvee U\} = \bigcup_{U \in \mathsf{Pow}(Y)} {\downarrow^B} \bigvee U$$

(cf. the extension J of a $j-$operator on a set-generated frame in [11]). Note that in the case of standard inductive definitions this operator has no role, as it reduces to identity.

The following proposition gathers together properties of c_L that we shall need in the following. Its proof is a simple exercise (due to the fact it is about classes, rather than just sets, in some cases Strong Collection is needed, cf. also [11]).

Proposition 4.1 *Let X, Y, X_i, for i in a set I, be subclasses of B. Then*

1. *Y is c_L-closed iff $Y = c_L Y$;*
2. *if Y is a set, $c_L Y = {\downarrow^B} \bigvee Y$;*
3. *$X \subseteq Y$ implies $c_L X \subseteq c_L Y$;*
4. *$Y \subseteq c_L Y$;*
5. *$c_L c_L Y = c_L Y$, so that $c_L Y$ is c_L-closed;*
6. *$X \subseteq c_L Y$ implies $c_L X \subseteq c_L Y$;*
7. *$c_L \bigcup_{i \in I} X_i = c_L \bigcup_{i \in I} c_L X_i$.*

For $Y \subseteq B$, define

$$\bar{\Gamma}_\Phi(Y) \equiv c_L\{b \in B \ \mid \ (\exists a)\ (b, a) \in \Phi \ \& \ {\downarrow^B} a \subseteq Y\}.$$

$\bar{\Gamma}_\Phi$ is clearly a monotone operator on classes.

We now construct the class that will give us the iterations we need. The following lemma and theorem are generalizations of the corresponding results for standard (non-abstract) inductive definitions (cf. Lemma 5.2 and Theorem 5.1 of [5]). Some extra work is needed to deal with the 'pointless' character of $\bigvee$-semilattices.

Let Φ be an abstract inductive definition on a $\bigvee$-semilattice L set-generated by a set B.

Lemma 4.2 *A class $J \subseteq \mathrm{V} \times B$ of ordered pairs exists such that, for every set $\alpha \in \mathrm{V}$,*

$$c_L J^\alpha = \bar{\Gamma}_\Phi(c_L J^{\in\alpha}).$$

Proof. A set $G \subseteq \mathrm{V} \times B$ will be called *good* (or an approximation to J) if, for any set α,

$$G^\alpha \subseteq \bar{\Gamma}_\Phi(c_L G^{\in\alpha}).$$

Define then

$$J = \bigcup \{G \mid G \text{ good }\}.$$

To prove $c_L J^\alpha \subseteq \bar{\Gamma}_\Phi(c_L J^{\in\alpha})$, it is enough to show that $J^\alpha \subseteq \bar{\Gamma}_\Phi(c_L J^{\in\alpha})$, since $\bar{\Gamma}_\Phi(Y)$ is c_L-closed for every Y. So let $b \in J^\alpha$. Then a good set G exists such that $b \in \bar{\Gamma}_\Phi(c_L G^{\in\alpha})$. Since $G^{\in\alpha} \subseteq J^{\in\alpha}$, also $c_L G^{\in\alpha} \subseteq c_L J^{\in\alpha}$, whence, by monotonicity of $\bar{\Gamma}_\Phi$, $b \in \bar{\Gamma}_\Phi(c_L J^{\in\alpha})$.

To prove the converse, let $y \in \bar{\Gamma}_\Phi(c_L J^{\in\alpha})$. There is then a set $U \in \mathsf{Pow}(B)$ with

$$U \subseteq \{b \in B \mid (\exists a)\ (b, a) \in \Phi \ \&\ {\downarrow^B} a \subseteq c_L J^{\in\alpha}\}, \text{ and } y \leq \bigvee U.$$

We shall prove that $U \subseteq J^\alpha$, so that $y \in c_L J^\alpha$. For $b \in U$ there is a pair $(b, a) \in \Phi$ such that ${\downarrow^B} a \subseteq c_L J^{\in\alpha}$. The last inclusion can be rewritten as

$$(\forall y' \in {\downarrow^B} a)\ (\exists W \in \mathsf{Pow}(J^{\in\alpha}))\ y' \leq \bigvee W.$$

By Strong Collection, a set K of subsets of $J^{\in\alpha}$ then exists such that

$$(\forall y' \in {\downarrow^B} a)\ (\exists W \in K)\ y' \leq \bigvee W.$$

Then,

$$(\forall y' \in {\downarrow^B} a)\ y' \leq \bigvee \bigcup K, \text{ and } \bigcup K \subseteq J^{\in\alpha}.$$

The latter expression can be reformulated as

$$(\forall d \in \bigcup K)(\exists x \in \alpha)\ d \in J^x, \text{ i.e.,}$$

$$(\forall d \in \bigcup K)(\exists G \in J)\ d \in G^{\in\alpha}.$$

By Strong Collection again, we then get a set Z of good sets such that

$$(\forall d \in \bigcup K)(\exists G \in Z)\ d \in G^{\in\alpha}.$$

Thus, $\bigcup K \subseteq (\bigcup Z)^{\in\alpha}$. Let then

$$G = \{(\alpha, b)\} \cup \bigcup Z.$$

As $\bigcup Z$ is a union of good sets, it is a good set too. Moreover, $\downarrow^B a \subseteq c_L G^{\in\alpha}$, indeed as seen $(\forall y' \in \downarrow^B a)\ y' \leq \bigvee \bigcup K$, and $\bigcup K \subseteq (\bigcup Z)^{\in\alpha} \subseteq G^{\in\alpha}$. Therefore, $b \in \bar{\Gamma}_\Phi(c_L G^{\in\alpha})$. Thus, G is a good set.

Now, since $\{(\alpha, b)\} \in G$, we have $b \in J^\alpha$. As this is true for every $b \in U$, and $y \leq \bigvee U$, we get $y \in c_L J^\alpha$, as wished.

Theorem 4.3 *Let Φ be an abstract inductive definition on a $\bigvee$-semilattice L set-generated by a set B. Then, the smallest $\Phi-$closed class $\mathcal{I}(\Phi)$ exists.*

Proof. Our goal is to prove that the class

$$c_L J^\infty = c_L \bigcup_{\alpha \in \mathrm{V}} J^\alpha$$

is the least $\Phi-$closed class. Let $(b, a) \in \Phi$, and $\downarrow^B a \subseteq c_L J^\infty$. Then, given $y \in \downarrow^B a$ there is a set U such that

$$U \in \mathsf{Pow}(J^\infty)\ \&\ y \leq \bigvee U.$$

So,

$$(\forall z \in U)(\exists \beta) z \in J^\beta.$$

By Collection, a set α exists such that

$$(\forall z \in U)(\exists \beta \in \alpha) z \in J^\beta.$$

Thus, $U \subseteq J^{\in\alpha}$, so that $y \in c_L J^{\in\alpha}$. So we have shown

$$(\forall y \in \downarrow^B a)(\exists \alpha) y \in c_L J^{\in\alpha}.$$

Applying again the Collection scheme, we then get a set K such that

$$(\forall y \in \downarrow^B a)(\exists \alpha \in K) y \in c_L J^{\in\alpha}.$$

Since $\alpha \in K$ implies $\alpha \subseteq \bigcup K$, using Proposition 4.1 we conclude $\downarrow^B a \subseteq c_L J^{\in \bigcup K}$. By definition of $\bar{\Gamma}_\Phi$, then $b \in \bar{\Gamma}_\Phi(c_L J^{\in \bigcup K})$. By Lemma 4.2,

$\bar\Gamma_\Phi(c_L J^{\in\bigcup K}) = c_L J^{\bigcup K}$, so that, as $J^{\bigcup K} \subseteq J^\infty$, we conclude that $c_L J^\infty$ is $\Phi-$closed.

It remains to show that $c_L J^\infty$ is the least $\Phi-$closed class. Let I be a $\Phi-$closed class, $I \subseteq B$. It is enough to prove that $J^\alpha \subseteq I$ for every set α, since then, as I is assumed c_L-closed, by Proposition 4.1 one has $c_L \bigcup_{\alpha\in \mathrm{V}} J^\alpha = c_L J^\infty \subseteq I$. We prove by Set Induction that, for every set α, $J^\alpha \subseteq I$: let α be a set. The inductive hypothesis gives $(\forall\beta \in \alpha) J^\beta \subseteq I$, i.e $J^{\in\alpha} \subseteq I$. By Proposition 4.1, then $c_L J^{\in\alpha} \subseteq c_L I = I$. By monotonicity of $\bar\Gamma_\Phi$, $\bar\Gamma_\Phi(c_L J^{\in\alpha}) \subseteq \bar\Gamma_\Phi(I) = I$, whence, by Lemma 4.2, $J^\alpha \subseteq c_L J^\alpha \subseteq I$.

It may be worth noting that, so far, we made no use of Exponentiation (let alone of the Subset Collection scheme), so that the above results in fact hold in CZF$^-$.

Corollary 4.4 (Tarski's fixed point theorem in CZF$^-$+ Sep) *The system CZF$^-$ augmented with the Separation Scheme proves that every monotone operator Γ on a set-generated $\bigvee$-semilattice L has a least fixed point.*

Proof. Let Φ_Γ be the inductive definition on L associated with Γ (see Section 3). By the previous theorem, the least $\Phi_\Gamma-$closed class $\mathcal{I}(\Phi_\Gamma) \subseteq B$ exists. Since B is a set, by Separation $\mathcal{I}(\Phi_\Gamma)$ is a set too. Then, by Propositions 3.1 and 3.2, $\bigvee \mathcal{I}(\Phi_\Gamma)$ is the least fixed point of Γ.

An elegant proof of Tarski's fixed point theorem making use of Separation, but no direct use of powerobjects is also presented in [14]. However, that proof appears to rely essentially on the assumption that L is small, a requirement that by Theorem 2.5 is hardly met in systems without powersets.

To obtain a predicative version of the above corollary, we generalize the concept of bounded inductive definition from [3] to abstract inductive definitions. A *bound* for an abstract inductive definition Φ is a set α such that, whenever $(b, a) \in \Phi$ there is $x \in \alpha$ such that the set $\downarrow^B a$ is an image of x. An abstract inductive definition Φ is *bounded* if

1. $\{b \in B \mid (b, a) \in \Phi\}$ is a set for every $a \in L$.

2. Φ has a bound.

Note that any abstract inductive definition that is a set is bounded. The following proposition generalizes the corresponding results for inductive definition on a set. Its proof is the obvious modification of the proof of [5, Proposition 5.6]. Exponentiation is used here for the first time in this paper.

Proposition 4.5 *Every bounded abstract inductive definition is local.*

Almost as obvious is the following generalization of [5, Proposition 5.3], provable in CZF$^-$.

Proposition 4.6 *If Φ is a local abstract inductive definition, $c_L J^\alpha$ and $c_L J^{\in\alpha}$ are sets for every α.*

Proof. We use induction on sets. Given $\alpha \in \mathrm{V}$, assume that for every $\beta \in \alpha$, $c_L J^\beta$ is a set. Since, by Proposition 4.1,

$$c_L J^{\in\alpha} \equiv c_L \bigcup_{\beta\in\alpha} J^\beta = c_L \bigcup_{\beta\in\alpha} c_L J^\beta,$$

and since $\bigcup_{\beta\in\alpha} c_L J^\beta$ is a set, $c_L J^{\in\alpha}$ is a set, too. Moreover, since for $Y \subseteq B$ a set,

$$\bar{\Gamma}_\Phi(Y) = \downarrow^B \Gamma_\Phi(\bigvee Y),$$

$\bar{\Gamma}_\Phi(Y)$ is a set for every set Y, as Φ is local and L is set-generated. By Lemma 4.2, $c_L J^\alpha = \bar{\Gamma}_\Phi(c_L J^{\in\alpha})$, so that $c_L J^\alpha$ is a set too. We can therefore conclude, by Set Induction, that for every $\alpha \in \mathrm{V}$, $c_L J^\alpha$, and so $c_L J^{\in\alpha}$, are sets.

In the case of a standard inductive definition Φ, one proves that if the inductive definition is bounded by a (weakly) regular bound, then CZF proves that $I(\Phi)$ is a set [5]. As a consequence, the system CZF + wREA proves that $I(\Phi)$ is a set whenever Φ is bounded. While the first fact does not seem to carry over to the present context, we have the following result.

Recall that a $\bigvee$-semilattice L set-generated by a set B is said to be *set-presented* [5] if a mapping $D : B \to \mathsf{Pow}(\mathsf{Pow}(B))$ is given with the property that $b \leq \bigvee U \iff (\exists W \in D(b)) W \subseteq U$, for every $b \in B, U \in \mathsf{Pow}(B)$.

Theorem 4.7 (CZF + uREA) *Let Φ be a bounded abstract inductive definition on a set-presented $\bigvee$-semilattice L. Then, the smallest $\Phi-$closed class $\mathcal{I}(\Phi)$ is a set.*

Proof. Let α be a bound for Φ, and let

$$S = \alpha \cup \{V : (\exists b \in B) V \in D(b)\} = \alpha \cup \bigcup Range(D).$$

By Replacement (that is a consequence of Strong Collection) and Union, S is a set. Then, by uREA, there is a set α' that is regular and $\bigcup -$closed, such that $S \subseteq \alpha'$.

We claim that

$$\mathcal{I}(\Phi) \equiv c_L J^{\infty} = c_L J^{\in\alpha'},$$

the latter class being a set by Propositions 4.5, 4.6.

Since $J^{\in\alpha'} \subseteq J^{\infty}$, $c_L J^{\in\alpha'} \subseteq c_L J^{\infty}$. So it remains to prove the converse, for which it suffices to show that $c_L J^{\in\alpha'}$ is $\Phi-$closed.

For $(b,a) \in \Phi$, assume $\downarrow^B a \subseteq c_L J^{\in\alpha'}$. Since α is a bound for Φ there is a set $Z \in \alpha$ and an onto mapping $f : Z \twoheadrightarrow \downarrow^B a$. So:

$$(\forall z \in Z) f(z) \in c_L J^{\in\alpha'},$$

i.e.,

$$(\forall z \in Z)(\exists U \in \mathsf{Pow}(J^{\in\alpha'})) f(z) \leq \bigvee U.$$

Since L is set-presented, for every $z \in Z$, there is $W \in D(f(z))$ such that $W \in \mathsf{Pow}(J^{\in\alpha'})$ and $f(z) \leq \bigvee W$. Therefore,

$$(\forall d \in W) d \in J^{\in\alpha'},$$

i.e.,

$$(\forall d \in W)(\exists \beta \in \alpha') d \in J^{\beta},$$

which implies

$$(\forall d \in W)(\exists \beta \in \alpha') d \in c_L J^{\beta}$$

(note that $c_L J^{\beta}$ is a set by Proposition 4.6, while J^{β} need not be). Now since, by construction, $W \in \alpha'$, and α' regular, there is a set $\gamma \in \alpha'$ such that

$$(\forall d \in W)(\exists \beta \in \gamma) d \in c_L J^{\beta}.$$

Thus, $W \subseteq \bigcup_{\beta\in\gamma} c_L J^{\beta}$, so that $f(z) \in c_L \bigcup_{\beta\in\gamma} c_L J^{\beta}$. By Proposition 4.1, $c_L \bigcup_{\beta\in\gamma} c_L J^{\beta} = c_L \bigcup_{\beta\in\gamma} J^{\beta} = c_L J^{\in\gamma}$. Thus, we have shown

$$(\forall z \in Z)(\exists \gamma \in \alpha') f(z) \in c_L J^{\in\gamma}.$$

Since $Z \in \alpha \subseteq \alpha'$, and α' regular, there is $\delta \in \alpha'$ such that

$$(\forall z \in Z)(\exists \gamma \in \delta) f(z) \in c_L J^{\in \gamma}.$$

Thus, $\downarrow^B a \subseteq \bigcup_{\gamma \in \delta} c_L J^{\in \gamma}$.

Now since α' union-closed, $\bigcup \delta \in \alpha'$. Also, since $J^{\in \gamma} \subseteq J^{\in \bigcup \delta}$ for every $\gamma \in \delta$, also $\bigcup_{\gamma \in \delta} c_L J^{\in \gamma} \subseteq c_L J^{\in \bigcup \delta}$. It follows that $\downarrow^B a \subseteq c_L J^{\in \bigcup \delta}$, so that, by Lemma 4.2, $b \in c_L J^{\bigcup \delta}$. Finally, as $\bigcup \delta \in \alpha'$, $c_L J^{\bigcup \delta} \subseteq c_L J^{\in \alpha'}$. Therefore $c_L J^{\in \alpha'}$ is $\Phi-$closed, as was to be proved.

As a corollary we get the *constructive Tarski's fixed point theorem*. Note that in classical set theory, or in a topos, every complete lattice is set-presented, and every monotone operator is obtained by a bounded abstract inductive definition (since in such a context any abstract inductive definition is a set).

Corollary 4.8 (CZF + uREA) *Let $\Gamma : L \to L$ be a monotone operator on a set-presented $\bigvee$-semilattice L. If $\Gamma = \Gamma_\Phi$ for Φ be a bounded abstract inductive definition on L, then Γ has a least fixed point.*

5 Conclusion

A version of Tarski's fixed point theorem for directed complete partial orders has been proved in recent years by Dito Pataraia. As for Tarski's proof, Pataraia's result uses topos valid but impredicative methods. Recall also that Tarski's theorem in its more general form states that a monotone map on a complete lattice has a complete lattice of fixed points, so that in particular it has a greatest fixed point. Constructive versions of these results along the lines of this work will be the subject of future investigation. This will also concern the possibility of applying constructive fixed-point theorems to the problem, raised in [1] (see also [7]), of obtaining predicative constructions of Frege structures.

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